

One-parameter statistical model for linear stochastic differential equation with time delay

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Abstract

Assume that we observe a stochastic process $(X(t))_{t \in [-r, T]}$, which satisfies the linear stochastic delay differential equation

$$dX(t) = \vartheta \int_{[-r, 0]} X(t+u) a(du) dt + dW(t), \quad t \geq 0,$$

where a is a finite signed measure on $[-r, 0]$. The local asymptotic properties of the likelihood function are studied. Local asymptotic normality is proved in case of $v_{\vartheta}^* < 0$, local asymptotic quadraticity is shown if $v_{\vartheta}^* = 0$, and, under some additional conditions, local asymptotic mixed normality or periodic local asymptotic mixed normality is valid if $v_{\vartheta}^* > 0$, where v_{ϑ}^* is an appropriately defined quantity. As an application, the asymptotic behaviour of the maximum likelihood estimator $\hat{\vartheta}_T$ of ϑ based on $(X(t))_{t \in [-r, T]}$ can be derived as $T \rightarrow \infty$.

1 Introduction

Consider the linear stochastic delay differential equation (SDDE)

$$(1.1) \quad \begin{cases} dX(t) = \vartheta \int_{[-r, 0]} X(t+u) a(du) dt + dW(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-r, 0], \end{cases}$$

where $r \in (0, \infty)$, $(W(t))_{t \in \mathbb{R}_+}$ is a standard Wiener process, $\vartheta \in \mathbb{R}$, and a is a finite signed measure on $[-r, 0]$ with $a \neq 0$, and $(X_0(t))_{t \in [-r, 0]}$ is a continuous process independent of $(W(t))_{t \in \mathbb{R}_+}$. The SDDE (1.1) can also be written in the integral form

$$(1.2) \quad \begin{cases} X(t) = X_0(0) + \vartheta \int_0^t \int_{[-r, 0]} X(s+u) a(du) ds + W(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-r, 0]. \end{cases}$$

2010 Mathematics Subject Classifications: 62B15, 62F12.

Key words and phrases: likelihood function; local asymptotic normality; local asymptotic mixed normality; periodic local asymptotic mixed normality; local asymptotic quadraticity; maximum likelihood estimator; stochastic differential equations; time delay.

Equation (1.1) is a special case of the affine stochastic delay differential equation

$$(1.3) \quad \begin{cases} dX(t) = \int_{-r}^0 X(t+u) a_{\vartheta}(du) dt + dW(t), & t \in \mathbb{R}_+, \\ X(t) = X_0(t), & t \in [-r, 0], \end{cases}$$

where $r > 0$, and for each $\vartheta \in \Theta$, a_{ϑ} is a finite signed measure on $[-r, 0]$, see Gushchin and K  chler [5]. In that paper local asymptotic normality (LAN) has been proved for stationary solutions. In Gushchin and K  chler [3], the special case of (1.3) has been studied with $r = 1$, $\Theta = \mathbb{R}^2$, and $a_{\vartheta} = \vartheta_1 \delta_0 + \vartheta_2 \delta_{-1}$ for $\vartheta = (\vartheta_1, \vartheta_2)$, where δ_x denotes the Dirac measure concentrated at $x \in \mathbb{R}$, and they described the local properties of the likelihood function for the whole parameter space \mathbb{R}^2 . In Benke and Pap [1], a special case has been studied, where $r = 1$ and a_{ϑ} is the Lebesgue measure multiplied by $\vartheta \in \mathbb{R}$.

In each of the above papers, LAN has been proved in case of $v_0(\vartheta) < 0$, where $v_0(\vartheta)$ is the real part of the right most characteristic roots of the corresponding deterministic homogeneous delay differential equation, see (1.8). It turns out that in case of equation (1.1), LAN holds whenever $v_{\vartheta}^* < 0$, where v_{ϑ}^* is defined in (3.1), see Theorem (3.1), but it can happen that $v_0(\vartheta) = 0$, see the example in Remark 3.4. Moreover, local asymptotic quadraticity (LAQ) is shown if $v_{\vartheta}^* = 0$, and, under some additional conditions, local asymptotic mixed normality (LAMN) or periodic local asymptotic mixed normality (PLAMN) is valid if $v_{\vartheta}^* > 0$, see Theorems 3.2 and 3.3. Note that in Theorems 3.2 and 3.3 we have $v_{\vartheta}^* = v_0(\vartheta)$, see Remark 3.4. The definition of LAN, LAQ, LAMN and PLAMN can be found in Le Cam and Yang [7] and Gushchin and K  chler [3].

The solution $(X^{(\vartheta)}(t))_{t \in \mathbb{R}_+}$ of (1.1) exists, is pathwise uniquely determined and can be represented as

$$(1.4) \quad \begin{aligned} X^{(\vartheta)}(t) &= x_{0,\vartheta}(t)X_0(0) + \vartheta \int_{[-r,0]} \int_u^0 x_{0,\vartheta}(t+u-s)X_0(s) ds a(du) \\ &\quad + \int_{[0,t]} W(t-s) dx_{0,\vartheta}(s), \quad t \in \mathbb{R}_+, \end{aligned}$$

where $(x_{0,\vartheta}(t))_{t \in [-r, \infty)}$ denotes the so-called fundamental solution of the deterministic homogeneous delay differential equation

$$(1.5) \quad \begin{cases} x(t) = x_0(0) + \vartheta \int_0^t \int_{[-r,0]} x(s+u) a(du) ds, & t \in \mathbb{R}_+, \\ x(t) = x_0(t), & t \in [-r, 0]. \end{cases}$$

with initial function

$$x_0(t) := \begin{cases} 0, & t \in [-r, 0), \\ 1, & t = 0, \end{cases}$$

which means that $x_{0,\vartheta}$ is absolutely continuous on \mathbb{R}_+ , $x_{0,\vartheta}(t) = 0$ for $t \in [-r, 0)$, $x_{0,\vartheta}(0) = 1$, and $\dot{x}_{0,\vartheta}(t) = \vartheta \int_{[-r,0]} x_{0,\vartheta}(t+u) a(du)$ for Lebesgue-almost all $t \in \mathbb{R}_+$. The

domain of integration in the last integral in (1.4) includes zero, i.e.,

$$\int_{[0,t]} W(t-s) dx_{0,\vartheta}(s) = W(t) + \int_{(0,t]} W(t-s) dx_{0,\vartheta}(s) = \int_0^t x_{0,\vartheta}(t-s) dW(s), \quad t \in \mathbb{R}_+.$$

In the trivial case of $\vartheta = 0$, we have $x_{0,0}(t) = 1$ for all $t \in \mathbb{R}_+$, and $X^{(0)}(t) = X_0(0) + W(t)$ for all $t \in \mathbb{R}_+$. The asymptotic behaviour of $x_{0,\vartheta}(t)$ as $t \rightarrow \infty$ is connected with the so-called characteristic function $h_\vartheta : \mathbb{C} \rightarrow \mathbb{C}$, given by

$$(1.6) \quad h_\vartheta(\lambda) := \lambda - \vartheta \int_{[-r,0]} e^{\lambda u} a(du), \quad \lambda \in \mathbb{C},$$

and the set Λ_ϑ of the (complex) solutions of the so-called characteristic equation for (1.5),

$$(1.7) \quad \lambda - \vartheta \int_{[-r,0]} e^{\lambda u} a(du) = 0.$$

Note that a complex number λ solves (1.7) if and only if $(e^{\lambda t})_{t \in [-r,\infty)}$ solves (1.5) with initial function $x_0(t) = e^{\lambda t}$, $t \in [-r, 0]$. We have $\Lambda_\vartheta \neq \emptyset$, $\overline{\Lambda_\vartheta} = \Lambda_\vartheta$, and Λ_ϑ consists of isolated points. Moreover, Λ_ϑ is countably infinite except the case where a is concentrated at 0, or $\vartheta = 0$. Further, for each $c \in \mathbb{R}$, the set $\{\lambda \in \Lambda_\vartheta : \operatorname{Re}(\lambda) \geq c\}$ is finite. In particular,

$$(1.8) \quad v_0(\vartheta) := \sup\{\operatorname{Re}(\lambda) : \lambda \in \Lambda_\vartheta\} < \infty.$$

For $\lambda \in \Lambda_\vartheta$, denote by $m_\vartheta(\lambda)$ the multiplicity of λ as a solution of (1.7).

The Laplace transform of $(x_{0,\vartheta}(t))_{t \in \mathbb{R}_+}$ is given by

$$\int_0^\infty e^{-\lambda t} x_{0,\vartheta}(t) dt = \frac{1}{h_\vartheta(\lambda)}, \quad \lambda \in \mathbb{C}, \quad \operatorname{Re}(\lambda) > v_0(\vartheta).$$

Based on the inverse Laplace transform and Cauchy's residue theorem, the following crucial lemma can be shown (see, e.g., Diekmann et al. [2, Lemma 5.1 and Theorem 5.4] or Gushchin and K  chler [4, Lemma 2.1]).

1.1 Lemma. *For each $\vartheta \in \mathbb{R}$ and each $c \in \mathbb{R}$, the fundamental solution $(x_{0,\vartheta}(t))_{t \in [-r,\infty)}$ of (1.5) can be represented in the form*

$$x_{0,\vartheta}(t) = \sum_{\substack{\lambda \in \Lambda_\vartheta \\ \operatorname{Re}(\lambda) \geq c}} \operatorname{Res}_{z=\lambda} \left(\frac{e^{zt}}{h_\vartheta(z)} \right) + \psi_{\vartheta,c}(t) = \sum_{\substack{\lambda \in \Lambda_\vartheta \\ \operatorname{Re}(\lambda) \geq c}} p_{\vartheta,\lambda}(t) e^{\lambda t} + \psi_{\vartheta,c}(t), \quad \text{as } t \rightarrow \infty,$$

where $\psi_{\vartheta,c} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous function with $\psi_{\vartheta,c}(t) = o(e^{ct})$ as $t \rightarrow \infty$, and for each $\vartheta \in \mathbb{R}$ and each $\lambda \in \Lambda_\vartheta$, $p_{\vartheta,\lambda}$ is a complex-valued polynomial of degree $m_\vartheta(\lambda) - 1$ with $p_{\vartheta,\bar{\lambda}} = \overline{p_{\vartheta,\lambda}}$. More exactly,

$$p_{\vartheta,\lambda}(t) = \sum_{\ell=0}^{m_\vartheta(\lambda)-1} \frac{A_{\vartheta,-1-\ell}(\lambda)}{\ell!} t^\ell,$$

where $A_{\vartheta,k}(\lambda)$, $k \in \{-m_{\vartheta}(\lambda), -m_{\vartheta}(\lambda) + 1, \dots\}$ denotes the coefficients of the Laurent's series of $1/h_{\vartheta}(z)$ at $z = \lambda$, i.e.,

$$\frac{1}{h_{\vartheta}(z)} = \sum_{k=-m_{\vartheta}(\lambda)}^{\infty} A_{\vartheta,k}(\lambda)(z - \lambda)^k$$

in a neighborhood of λ .

As a consequence, for any $c > v_0(\vartheta)$, we have $x_{0,\vartheta}(t) = O(e^{ct})$, as $t \rightarrow \infty$. In particular, $(x_{0,\vartheta}(t))_{t \in \mathbb{R}_+}$ is square integrable if (and only if, see Gushchin and K  chler [4]) $v_0(\vartheta) < 0$.

2 Radon–Nikodym derivatives

From this section, we will consider the SDDE (1.1) with fixed continuous initial process $(X_0(t))_{t \in [-r,0]}$. Further, for all $T \in \mathbb{R}_{++}$, let $\mathbb{P}_{\vartheta,T}$ be the probability measure induced by $(X^{(\vartheta)}(t))_{t \in [-r,T]}$ on $(C([-r,T]), \mathcal{B}(C([-r,T])))$. In order to calculate Radon–Nikodym derivatives $\frac{d\mathbb{P}_{\theta,T}}{d\mathbb{P}_{\vartheta,T}}$ for certain $\theta, \vartheta \in \mathbb{R}$, we need the following statement, which can be derived from formula (7.139) in Section 7.6.4 of Liptser and Shiryaev [8].

2.1 Lemma. *Let $\theta, \vartheta \in \mathbb{R}$. Then for all $T \in \mathbb{R}_{++}$, the measures $\mathbb{P}_{\theta,T}$ and $\mathbb{P}_{\vartheta,T}$ are absolutely continuous with respect to each other, and*

$$\begin{aligned} \log \frac{d\mathbb{P}_{\theta,T}}{d\mathbb{P}_{\vartheta,T}}(X^{(\vartheta)}|_{[-r,T]}) &= (\theta - \vartheta) \int_0^T Y^{(\vartheta)}(t) dX^{(\vartheta)}(t) - \frac{1}{2}(\theta^2 - \vartheta^2) \int_0^T Y^{(\vartheta)}(t)^2 dt \\ &= (\theta - \vartheta) \int_0^T Y^{(\vartheta)}(t) dW(t) - \frac{1}{2}(\theta - \vartheta)^2 \int_0^T Y^{(\vartheta)}(t)^2 dt \end{aligned}$$

with

$$Y^{(\vartheta)}(t) := \int_{[-r,0]} X^{(\vartheta)}(t+u) a(du), \quad t \in \mathbb{R}_+.$$

In order to investigate local asymptotic properties of the family

$$(2.1) \quad (\mathcal{E}_T)_{T \in \mathbb{R}_{++}} := (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)), \{\mathbb{P}_{\vartheta,T} : \vartheta \in \mathbb{R}\})_{T \in \mathbb{R}_{++}}$$

of statistical experiments, we derive the following corollary.

2.2 Corollary. *For each $\vartheta \in \mathbb{R}$, $T \in \mathbb{R}_{++}$, $r_{\vartheta,T} \in \mathbb{R}$ and $h_T \in \mathbb{R}$, we have*

$$\log \frac{d\mathbb{P}_{\vartheta+r_{\vartheta,T}h_T,T}}{d\mathbb{P}_{\vartheta,T}}(X^{(\vartheta)}|_{[-r,T]}) = h_T \Delta_{\vartheta,T} - \frac{1}{2} h_T^2 J_{\vartheta,T},$$

with

$$\Delta_{\vartheta,T} := r_{\vartheta,T} \int_0^T Y^{(\vartheta)}(t) dW(t), \quad J_{\vartheta,T} := r_{\vartheta,T}^2 \int_0^T Y^{(\vartheta)}(t)^2 dt.$$

3 Local asymptotics of likelihood ratios

For each $\lambda \in \Lambda_\vartheta$, denote by $\tilde{m}_\vartheta(\lambda)$ the degree of the complex-valued polynomial

$$P_{\vartheta,\lambda}(t) := \sum_{\ell=0}^{m_\vartheta(\lambda)-1} c_{\vartheta,\lambda,\ell} t^\ell$$

with

$$c_{\vartheta,\lambda,\ell} := \frac{1}{\ell!} \int_{[-r,0]} \operatorname{Res}_{z=\lambda} \left(\frac{(z-\lambda)^\ell e^{zu}}{h_\vartheta(z)} \right) a(du) = \frac{1}{\ell!} \sum_{j=0}^{m_\vartheta(\lambda)-1-\ell} \frac{A_{\vartheta,-j-1-\ell}(\lambda)}{j!} \int_{[-r,0]} u^j e^{\lambda u} a(du),$$

where the degree of the zero polynomial is defined to be $-\infty$. Put

$$(3.1) \quad v_\vartheta^* := \sup\{\operatorname{Re}(\lambda) : \lambda \in \Lambda_\vartheta, \tilde{m}_\vartheta(\lambda) \geq 0\}, \quad m_\vartheta^* := \max\{\tilde{m}_\vartheta(\lambda) : \lambda \in \Lambda_\vartheta, \operatorname{Re}(\lambda) = v_\vartheta^*\},$$

where $\sup \emptyset := -\infty$ and $\max \emptyset := -\infty$.

3.1 Theorem. *If $\vartheta \in \mathbb{R}$ with $v_\vartheta^* < 0$, then the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ of statistical experiments given in (2.1) is LAN at ϑ with scaling $r_{\vartheta,T} = T^{-1/2}$, $T \in \mathbb{R}_{++}$, and with*

$$J_\vartheta = \int_0^\infty \left(\int_{[-r,0]} x_{0,\vartheta}(t+u) a(du) \right)^2 dt.$$

Particularly, if $a([-r,0]) = 0$, then $v_0^ = -\infty$, $m_0^* = -\infty$, and the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ of statistical experiments given in (2.1) is LAN at 0 with scaling $r_{0,T} = T^{-1/2}$, $T \in \mathbb{R}_{++}$, and with*

$$J_0 = \int_0^r a([-t,0])^2 dt.$$

3.2 Theorem. *If $\vartheta \in \mathbb{R}$ with $v_\vartheta^* = 0$, then the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ of statistical experiments given in (2.1) is LAQ at ϑ with scaling $r_{\vartheta,T} = T^{-m_\vartheta^*-1}$ and with*

$$\begin{aligned} \Delta_\vartheta &= \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} \int_0^1 \mathcal{Z}_{\operatorname{Im}(\lambda),m_\vartheta^*}(s) d\overline{\mathcal{Z}_{\operatorname{Im}(\lambda),0}(s)}, \\ J_\vartheta &= \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} |c_{\vartheta,\lambda,m_\vartheta^*}|^2 \int_0^1 |\mathcal{Z}_{\operatorname{Im}(\lambda),m_\vartheta^*}(s)|^2 ds, \end{aligned}$$

with

$$\mathcal{Z}_{\varphi,0} := \begin{cases} \mathcal{W}, & \text{if } \varphi = 0, \\ \frac{1}{\sqrt{2}}(\mathcal{W}_{\varphi,\operatorname{Re}} + i\mathcal{W}_{\varphi,\operatorname{Im}}), & \text{if } \varphi \in \mathbb{R}_{++}, \\ \overline{\mathcal{Z}_{-\varphi,0}}, & \text{if } \varphi \in \mathbb{R}_{--}, \end{cases}$$

where $(\mathcal{W}(s))_{s \in [0,1]}$, $(\mathcal{W}_{\varphi, \text{Re}}(s))_{s \in [0,1]}$ and $(\mathcal{W}_{\varphi, \text{Im}}(s))_{s \in [0,1]}$, $\varphi \in \mathbb{R}_{++}$, are independent standard Wiener processes, and

$$\mathcal{Z}_{\varphi, \ell}(s) := \int_0^s (s-u)^\ell d\mathcal{Z}_{\varphi, 0}(u), \quad s \in [0, 1], \quad \varphi \in \mathbb{R}, \quad \ell \in \mathbb{N}.$$

Particularly, if $a([-r, 0]) \neq 0$, then $v_0^* = 0$, $m_0^* = 0$, and the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ of statistical experiments given in (2.1) is LAQ at 0 with scaling $r_{0,T} = T^{-1}$, $T \in \mathbb{R}_{++}$, and with

$$\Delta_0 = a([-r, 0]) \int_0^1 \mathcal{W}(s) d\mathcal{W}(s), \quad J_0 = a([-r, 0])^2 \int_0^1 \mathcal{W}(s)^2 ds.$$

Note that Δ_ϑ is almost surely real-valued, since

$$c_{\vartheta, \bar{\lambda}, m_\vartheta^*} \int_0^1 \mathcal{Z}_{\text{Im}(\bar{\lambda}), m_\vartheta^*}(s) d\overline{\mathcal{Z}_{\text{Im}(\bar{\lambda}), 0}(s)} = \overline{c_{\vartheta, \lambda, m_\vartheta^*}} \int_0^1 \overline{\mathcal{Z}_{\text{Im}(\lambda), m_\vartheta^*}(s)} d\mathcal{Z}_{\text{Im}(\lambda), 0}(s), \quad \lambda \in \Lambda_\vartheta.$$

3.3 Theorem. Let $\vartheta \in \mathbb{R}$ with $v_\vartheta^* > 0$. If

$$H_\vartheta := \{\text{Im}(\lambda) : \lambda \in \Lambda_\vartheta \cap (v_\vartheta^* + i\mathbb{R}_{++}), \tilde{m}_\vartheta(\lambda) = m_\vartheta^*\} \neq \emptyset,$$

and the numbers in H_ϑ have a common divisor D_ϑ , then the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ of statistical experiments given in (2.1) is PLAMN at ϑ with period $\frac{2\pi}{D_\vartheta}$, with scaling $r_{\vartheta,T} = T^{-m_\vartheta^*} e^{-v_\vartheta^* T}$, $T \in \mathbb{R}_{++}$, and with

$$\Delta_\vartheta(d) = Z \sqrt{J_\vartheta(d)}, \quad J_\vartheta(d) = \int_0^\infty e^{-2v_\vartheta^* t} \text{Re} \left(\sum_{\substack{\lambda \in \Lambda_\vartheta \cap (v_\vartheta^* + i\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} c_{\vartheta, \lambda, m_\vartheta^*} U_\lambda^{(\vartheta)} e^{i(d-t) \text{Im}(\lambda)} \right)^2 dt,$$

for $d \in [0, \frac{2\pi}{D_\vartheta})$, where

$$U_\lambda^{(\vartheta)} := X_0(0) + v_\vartheta^* \int_{[-r, 0]} \int_u^0 e^{-\lambda(s-u)} X_0(s) ds a(du) + \int_0^\infty e^{-\lambda s} dW(s), \quad \lambda \in \mathbb{C},$$

and Z is a standard normally distributed random variable independent of $(X_0(t))_{t \in [-r, 0]}$ and $(W(t))_{t \in \mathbb{R}_+}$.

If $H_\vartheta = \emptyset$, then the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ of statistical experiments given in (2.1) is LAMN at ϑ with scaling $r_{\vartheta,T} = T^{-m_\vartheta^*} e^{-v_\vartheta^* T}$, $T \in \mathbb{R}_{++}$, and with

$$\Delta_\vartheta = Z \sqrt{J_\vartheta}, \quad J_\vartheta = \frac{c_{\vartheta, v_\vartheta^*, m_\vartheta^*}^2}{2v_\vartheta^*} (U_{v_\vartheta^*}^{(\vartheta)})^2.$$

3.4 Remark. According to the definition of $v_0(\vartheta)$ and v_ϑ^* , we obtain $v_0(\vartheta) \geq v_\vartheta^*$. The aim of the following discussion is to show that $v_0(\vartheta) > v_\vartheta^*$ if and only if $\{\lambda \in \Lambda_\vartheta : \text{Re}(\lambda) \geq 0\} = \{0\}$ and $P_{\vartheta, 0} = 0$ (and hence $v_0(\vartheta) = 0$ and $v_\vartheta^* < 0$). Indeed, if $v_0(\vartheta) > v_\vartheta^*$ then for all $\lambda_0 \in \Lambda_\vartheta$ with $\text{Re}(\lambda_0) > v_\vartheta^*$ we have $P_{\vartheta, \lambda_0} = 0$, implying

$$c_{\vartheta, \lambda_0, m_\vartheta(\lambda_0)-1} = \frac{1}{(m_\vartheta(\lambda_0) - 1)!} A_{\vartheta, -m_\vartheta(\lambda_0)}(\lambda_0) \int_{[-r, 0]} e^{\lambda_0 u} a(du) = 0.$$

Here $A_{\vartheta, -m_{\vartheta}(\lambda_0)}(\lambda) \neq 0$, since it is the leading coefficient of the polynomial p_{ϑ, λ_0} of degree $m_{\vartheta}(\lambda_0) - 1$, hence $c_{\vartheta, \lambda_0, m_{\vartheta}(\lambda_0)-1} = 0$ yields $\int_{[-r, 0]} e^{\lambda_0 u} a(du) = 0$. Taking into account of the characteristic equation, we get $\lambda_0 = 0$, hence $\{\lambda \in \Lambda_{\vartheta} : \operatorname{Re}(\lambda) \geq v_{\vartheta}^*\} = \{0\}$. Clearly, this yields also $v_0(\vartheta) = 0$ and $v_{\vartheta}^* < 0$, and hence $\{\lambda \in \Lambda_{\vartheta} : \operatorname{Re}(\lambda) \geq 0\} = \{0\}$. Conversely, if $\{\lambda \in \Lambda_{\vartheta} : \operatorname{Re}(\lambda) \geq 0\} = \{0\}$ and $P_{\vartheta, 0} = 0$, then, by definition, $v_0(\vartheta) = 0$ and $v_{\vartheta}^* < 0$.

In particular, if $\{\lambda \in \Lambda_{\vartheta} : \operatorname{Re}(\lambda) \geq 0\} = \{0\}$ and $m_{\vartheta}(0) = 1$, then $v_0(\vartheta) > v_{\vartheta}^*$ is equivalent to $a([-r, 0]) = 0$, since $P_{\vartheta, 0} = c_{\vartheta, 0, 0} = \int_{[-r, 0]} e^{\lambda_0 u} a(du) = a([-r, 0])$.

An example for this situation is, when $r = 2\pi$, $a(du) = \sin(u)du$ and $\vartheta \in (0, \frac{1}{\pi})$. This can be derived, applying usual methods (e.g., argument principle in complex analysis and the existence of local inverses of holomorphic functions), see, e.g., Reiß [9].

3.5 Remark. Using these results, we can give the asymptotic behaviour of the maximum likelihood estimator of ϑ based on the observation $(X(t))_{t \in [-1, T]}$ for some fixed $T > 0$, see, e.g., Benke and Pap [1, Section 5].

4 Proofs

For each $\vartheta \in \mathbb{R}$ and each $t \in [r, \infty)$, by (1.4), we have

$$\begin{aligned} Y^{(\vartheta)}(t) &= X_0(0) \int_{[-r, 0]} x_{0, \vartheta}(t+u) du + \int_{[-r, 0]} \int_{[0, t+u]} W(t+u-s) dx_{0, \vartheta}(s) a(du) \\ &\quad + \vartheta \int_{[-r, 0]} \int_{[-r, 0]} \int_v^0 x_{0, \vartheta}(t+u+v-s) X_0(s) ds a(dv) a(du). \end{aligned}$$

Here we have

$$\begin{aligned} &\int_{[-r, 0]} \int_{[-r, 0]} \int_v^0 x_{0, \vartheta}(t+u+v-s) X_0(s) ds a(dv) a(du) \\ &= \int_{[-r, 0]} \int_{[-r, 0]} \int_v^0 x_{0, \vartheta}(t+u+v-s) X_0(s) ds a(du) a(dv) \\ &= \int_{[-r, 0]} \int_v^0 X_0(s) \int_{[-r, 0]} x_{0, \vartheta}(t+u+v-s) a(du) ds a(dv), \end{aligned}$$

and

$$\begin{aligned} &\int_{[-r, 0]} \int_0^{t+u} x_{0, \vartheta}(t+u-s) dW(s) a(du) \\ &= \int_0^{t-r} \int_{[-r, 0]} x_{0, \vartheta}(t+u-s) a(du) dW(s) + \int_{t-r}^t \int_{[s-t, 0]} x_{0, \vartheta}(t+u-s) a(du) dW(s) \\ &= \int_0^t \int_{[-r, 0]} x_{0, \vartheta}(t+u-s) a(du) dW(s), \end{aligned}$$

since $t \in [r, \infty)$, $s \in [t - r, t]$ and $u \in [-r, s - t]$ imply $t + u - s \in [-r, 0]$, and hence $x_{0,\vartheta}(t + u - s) = 0$. Consequently, the process $(Y^{(\vartheta)}(t))_{t \in [r, \infty)}$ has a representation

$$(4.1) \quad Y^{(\vartheta)}(t) = y_{\vartheta}(t)X_0(0) + \vartheta \int_{[-r,0]} \int_v^0 y_{\vartheta}(t + v - s)X_0(s) ds a(dv) + \int_0^t y_{\vartheta}(t - s) dW(s)$$

for $t \in [r, \infty)$ with

$$y_{\vartheta}(t) := \int_{[-r,0]} x_{0,\vartheta}(t + u) a(du), \quad t \in \mathbb{R}_+.$$

Applying Lemma 1.1, we obtain

$$y_{\vartheta}(t) = \sum_{\substack{\lambda \in \Lambda_{\vartheta} \\ \operatorname{Re}(\lambda) \geq c}} \int_{[-r,0]} \operatorname{Res}_{z=\lambda} \left(\frac{e^{z(t+u)}}{h_{\vartheta}(z)} \right) a(du) + \int_{[-r,0]} \psi_{\vartheta,c}(t + u) a(du).$$

Here we have

$$\int_{[-r,0]} \psi_{\vartheta,c}(t + u) a(du) = o(e^{ct}) \quad \text{as } t \rightarrow \infty.$$

Indeed,

$$\lim_{t \rightarrow \infty} e^{-ct} \int_{[-r,0]} \psi_{\vartheta,c}(t + u) a(du) = \lim_{t \rightarrow \infty} \int_{[-r,0]} [e^{-c(t+u)} \psi_{\vartheta,c}(t + u)] e^{cu} a(du) = 0,$$

since a is a finite signed measure on $[-r, 0]$. Moreover,

$$\begin{aligned} \operatorname{Res}_{z=\lambda} \left(\frac{e^{z(t+u)}}{h_{\vartheta}(z)} \right) &= e^{\lambda(t+u)} \sum_{k=-m_{\vartheta}(\lambda)}^{-1} \frac{A_{\vartheta,k}(\lambda)}{(-1-k)!} (t+u)^{-1-k} \\ &= e^{\lambda(t+u)} \sum_{k=-m_{\vartheta}(\lambda)}^{-1} A_{\vartheta,k}(\lambda) \sum_{\ell=0}^{-1-k} \frac{t^{\ell} u^{-1-k-\ell}}{\ell! (-1-k-\ell)!} \\ &= e^{\lambda(t+u)} \sum_{\ell=0}^{m_{\vartheta}(\lambda)-1} \frac{t^{\ell}}{\ell!} \sum_{k=-m_{\vartheta}(\lambda)}^{-1-\ell} \frac{A_{\vartheta,k}(\lambda)}{(-1-k-\ell)!} u^{-1-k-\ell} \\ &= e^{\lambda t} \sum_{\ell=0}^{m_{\vartheta}(\lambda)-1} \frac{t^{\ell}}{\ell!} \operatorname{Res}_{z=\lambda} \left(\frac{(z-\lambda)^{\ell} e^{zu}}{h_{\vartheta}(z)} \right). \end{aligned}$$

Consequently, we obtain for each $\vartheta \in \mathbb{R}$ and each $c \in \mathbb{R}$, the representation

$$(4.2) \quad y_{\vartheta}(t) = \sum_{\substack{\lambda \in \Lambda_{\vartheta} \\ \operatorname{Re}(\lambda) \geq c}} P_{\vartheta,\lambda}(t) e^{\lambda t} + \Psi_{\vartheta,c}(t) \quad \text{as } t \rightarrow \infty,$$

where $\Psi_{\vartheta,c} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous function with $\Psi_{\vartheta,c}(t) = o(e^{ct})$ as $t \rightarrow \infty$. Hence we need to analyse the asymptotic behavior of the right hand side of (4.1) as $T \rightarrow \infty$, replacing $y_{\vartheta}(t)$ by $P_{\vartheta,\lambda}(t)e^{\lambda t}$.

First we derive a good estimate for the second term of the right hand side of (4.1).

4.1 Lemma. Let $(y(t))_{t \in \mathbb{R}_+}$ be a continuous deterministic function. Let a be a signed measure on $[-r, 0]$. Put

$$I(t) := \int_{[-r, 0]} \int_u^0 y(t+u-s) X_0(s) \, ds \, a(du), \quad t \in [r, \infty).$$

Then for each $T \in [r, \infty)$,

$$(4.3) \quad I(T)^2 \leq |a|([-r, 0]) \int_{-r}^0 X_0(s)^2 \, ds \int_0^T y(v)^2 \, dv \leq \|a\| \int_{-r}^0 X_0(s)^2 \, ds \int_0^T y(v)^2 \, dv,$$

$$(4.4) \quad \begin{aligned} \int_r^T I(t)^2 \, dt &\leq \int_{[-r, 0]} (-u) |a|(du) \int_{-r}^0 X_0(s)^2 \, ds \int_0^T y(v)^2 \, dv \\ &\leq r \|a\| \int_{-r}^0 X_0(s)^2 \, ds \int_0^T y(v)^2 \, dv, \end{aligned}$$

where $|a|$ and $\|a\| := |a|([-r, 0])$ denotes the variation and the total variation of the signed measure a , respectively.

Proof. For each $t \in [r, \infty)$, by Fubini's theorem,

$$I(t) = \int_{-r}^0 X_0(s) \int_{[-r, s]} y(t+u-s) \, a(du) \, ds.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} I(t)^2 &\leq \int_{-r}^0 X_0(s)^2 \, ds \int_{-r}^0 \left(\int_{[-r, s]} y(t+u-s) \, a(du) \right)^2 \, ds \\ &\leq \int_{-r}^0 X_0(s)^2 \, ds \int_{-r}^0 \int_{[-r, s]} y(t+u-s)^2 |a|(du) \, ds, \end{aligned}$$

where, by Fubini's theorem,

$$\begin{aligned} \int_{-r}^0 \int_{[-r, s]} y(t+u-s)^2 |a|(du) \, ds &= \int_{[-r, 0]} \int_u^0 y(t+u-s)^2 \, ds |a|(du) \\ &= \int_{[-r, 0]} \int_{t+u}^t y(v)^2 \, dv |a|(du) \leq \int_{[-r, 0]} |a|(du) \int_0^t y(v)^2 \, dv, \end{aligned}$$

hence we obtain (4.3). Moreover,

$$\int_r^T I(t)^2 \, dt \leq \int_{-r}^0 X_0(s)^2 \, ds \int_r^T \int_{-r}^0 \int_{[-r, s]} y(t+u-s)^2 |a|(du) \, ds \, dt,$$

where, by Fubini's theorem,

$$\begin{aligned}
\int_r^T \int_{-r}^0 \int_{[-r,s]} y(t+u-s)^2 |a|(du) ds dt &= \int_{-r}^0 \int_{[-r,s]} \int_r^T y(t+u-s)^2 dt |a|(du) ds \\
&= \int_{-r}^0 \int_{[-r,s]} \int_{r+u-s}^{T+u-s} y(v)^2 dv |a|(du) ds \leq \int_0^T y(v)^2 dv \int_{-r}^0 \int_{[-r,s]} |a|(du) ds \\
&= \int_0^T y(v)^2 dv \int_{[-r,0]} \int_u^0 ds |a|(du) = \int_0^T y(v)^2 dv \int_{[-r,0]} (-u) |a|(du),
\end{aligned}$$

hence we obtain (4.4). \square

4.2 Lemma. Let $(y(t))_{t \in \mathbb{R}_+}$ be a continuous deterministic function with $\int_0^\infty y(t)^2 dt < \infty$. Let $\vartheta \in \mathbb{R}$. Suppose that $(Y(t))_{t \in \mathbb{R}_+}$ is a continuous stochastic process such that

$$(4.5) \quad Y(t) = y(t)X_0(0) + \vartheta \int_{[-r,0]} \int_v^0 y(t+v-s)X_0(s) ds a(dv) + \int_0^t y(t-s) dW(s)$$

for $t \in [r, \infty)$. Then

$$(4.6) \quad \frac{1}{T} \int_0^T Y(t) dt \xrightarrow{\mathbb{P}} 0 \quad \text{as } T \rightarrow \infty,$$

$$(4.7) \quad \frac{1}{T} \int_0^T Y(t)^2 dt \xrightarrow{\mathbb{P}} \int_0^\infty y(t)^2 dt \quad \text{as } T \rightarrow \infty.$$

Proof. Applying Lemma 4.3 in Gushchin and K  chler [3] for the special case $X_0(s) = 0$, $s \in [-1, 0]$, we obtain

$$\begin{aligned}
\frac{1}{T} \int_0^T \int_0^t y(t-s) dW(s) dt &\xrightarrow{\mathbb{P}} 0 \quad \text{as } T \rightarrow \infty, \\
\frac{1}{T} \int_0^T \left(\int_0^t y(t-s) dW(s) \right)^2 dt &\xrightarrow{\mathbb{P}} \int_0^\infty y(t)^2 dt \quad \text{as } T \rightarrow \infty.
\end{aligned}$$

We have

$$\frac{1}{T} \int_0^T Y(t) dt = \frac{1}{T} \int_0^r Y(t) dt + X_0(0)I_1(T) + \frac{\vartheta}{T} \int_r^T Z(t) dt + \frac{1}{T} \int_r^T \int_0^t y(t-s) dW(s) dt$$

for $T \in \mathbb{R}_+$ with

$$\begin{aligned}
I_1(T) &:= \frac{1}{T} \int_r^T y(t) dt, \quad T \in \mathbb{R}_+, \\
(4.8) \quad Z(t) &:= \int_{[-r,0]} \int_u^0 y(t+u-s)X_0(s) ds a(du), \quad t \in [r, \infty).
\end{aligned}$$

By the Cauchy–Schwarz inequality and by (4.4),

$$|I_1(T)| \leq \sqrt{\int_r^T \frac{1}{T^2} dt \int_r^T y(t)^2 dt} \leq \sqrt{\frac{1}{T} \int_0^\infty y(t)^2 dt} \rightarrow 0,$$

$$\left| \frac{1}{T} \int_r^T Z(t) dt \right| \leq \sqrt{\frac{1}{T} \int_r^T Z(t)^2 dt} \leq \sqrt{\frac{r\|a\|}{T} \int_{-r}^0 X_0(s)^2 ds \int_0^\infty y(v)^2 dv} \xrightarrow{\text{a.s.}} 0$$

as $T \rightarrow \infty$, hence we obtain (4.6). Moreover,

$$\frac{1}{T} \int_0^T Y(t)^2 dt = \frac{1}{T} \int_0^r Y(t)^2 dt + I_2(T) + 2I_3(T) + \frac{1}{T} \int_r^T \left(\int_0^t y(t-s) dW(s) \right)^2 dt$$

for $T \in \mathbb{R}_+$, with

$$I_2(T) := \frac{1}{T} \int_r^T (y(t)X_0(0) + \vartheta Z(t))^2 dt,$$

$$I_3(T) := \frac{1}{T} \int_r^T (y_i(t)X_0(0) + \vartheta Z_i(t)) \left(\int_0^t y_i(t-s) dW(s) \right) dt.$$

Again by (4.4),

$$\begin{aligned} 0 &\leq I_2(T) \leq \frac{1}{T} \int_r^T 2(y(t)^2 X_0(0)^2 + \vartheta^2 Z(t)^2) dt \\ &\leq \frac{2X_0(0)^2}{T} \int_0^\infty y(t)^2 dt + \frac{2}{T} \vartheta^2 r \|a\| \int_{-r}^0 X_0(s)^2 ds \int_0^\infty y(v)^2 dv \xrightarrow{\text{a.s.}} 0 \end{aligned}$$

as $T \rightarrow \infty$, and

$$\begin{aligned} |I_3(T)| &\leq \frac{2}{T} \sqrt{\int_r^T (y(t)X_0(0) + \vartheta Z(t))^2 dt \int_r^T \left(\int_0^t y(t-s) dW(s) \right)^2 dt} \\ &= 2 \sqrt{\frac{I_2(T)}{T} \int_r^T \left(\int_0^t y(t-s) dW(s) \right)^2 dt} \xrightarrow{\mathbb{P}} 0 \quad \text{as } T \rightarrow \infty, \end{aligned}$$

hence we obtain (4.7). \square

4.3 Lemma. Let $y_\ell(t) = t^{\alpha_\ell} \operatorname{Re}(c_\ell e^{\lambda_\ell t})$, $t \in \mathbb{R}_+$, $\ell \in \{1, 2\}$, with some $\alpha_\ell \in \mathbb{Z}_+$, $c_\ell, \lambda_\ell \in \mathbb{C}$ with $\operatorname{Re}(\lambda_\ell) \in \mathbb{R}_{++}$, $\ell \in \{1, 2\}$. Let $\vartheta \in \mathbb{R}$. Suppose that $(Y_\ell(t))_{t \in \mathbb{R}_+}$, $\ell \in \{1, 2\}$, are continuous stochastic processes admitting representation (4.5) on $[r, \infty)$ with $y = y_\ell$, $\ell \in \{1, 2\}$, respectively. Then

$$(4.9) \quad t^{-\alpha_1} e^{-t \operatorname{Re}(\lambda_1)} Y_1(t) - \operatorname{Re}(c_1 U_{\lambda_1}^{(\vartheta)} e^{it \operatorname{Im}(\lambda_1)}) \xrightarrow{\text{a.s.}} 0, \quad \text{as } t \rightarrow \infty,$$

and

$$\begin{aligned} (4.10) \quad & T^{-\alpha_1 - \alpha_2} e^{-T \operatorname{Re}(\lambda_1 + \lambda_2)} \int_0^T Y_1(t) Y_2(t) dt \\ & - \int_0^\infty e^{-t \operatorname{Re}(\lambda_1 + \lambda_2)} \operatorname{Re}(c_1 U_{\lambda_1}^{(\vartheta)} e^{i(T-t) \operatorname{Im}(\lambda_1)}) \operatorname{Re}(c_2 U_{\lambda_2}^{(\vartheta)} e^{i(T-t) \operatorname{Im}(\lambda_2)}) dt \xrightarrow{\text{a.s.}} 0, \end{aligned}$$

as $T \rightarrow \infty$. Particularly, if $c_1, c_2 \in \mathbb{R}$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{++}$, then

$$t^{-\alpha_1} e^{-\lambda_1 t} Y_1(t) \xrightarrow{\text{a.s.}} c_1 U_{\lambda_1}^{(\vartheta)}, \quad \text{as } t \rightarrow \infty,$$

$$T^{-\alpha_1 - \alpha_2} e^{-T(\lambda_1 + \lambda_2)} \int_0^T Y_1(t) Y_2(t) dt \xrightarrow{\text{a.s.}} \frac{c_1 c_2}{\lambda_1 + \lambda_2} U_{\lambda_1}^{(\vartheta)} U_{\lambda_1}^{(\vartheta)}, \quad \text{as } T \rightarrow \infty.$$

Proof. Note that

$$t^{-\alpha_1} e^{-t \operatorname{Re}(\lambda_1)} Y_1(t) - \operatorname{Re}(c_1 U_{\lambda_1}^{(\vartheta)} e^{it \operatorname{Im}(\lambda_1)}) = -I_1(t) + I_2(t) - I_3(t), \quad t \in [r, \infty),$$

with

$$I_1(t) := \vartheta \int_{[-r, 0]} \int_v^0 \left[1 - \left(1 - \frac{s-v}{t} \right)^{\alpha_1} \right] \operatorname{Re}(c_1 e^{it \operatorname{Im}(\lambda_1) - \lambda_1(s-v)}) X_0(s) ds a(dv),$$

$$I_2(t) := \int_0^t \left[\left(1 - \frac{s}{t} \right)^{\alpha_1} - 1 \right] \operatorname{Re}(c_1 e^{it \operatorname{Im}(\lambda_1) - \lambda_1 s}) dW(s),$$

$$I_3(t) := \int_t^\infty \operatorname{Re}(c_1 e^{it \operatorname{Im}(\lambda_1) - \lambda_1 s}) dW(s).$$

By the dominated convergence theorem, $I_1(t) \xrightarrow{\text{a.s.}} 0$ as $t \rightarrow \infty$. Moreover,

$$I_2(t) = \sum_{k=1}^{\alpha_1} (-1)^k \binom{\alpha_1}{k} t^{-k} \int_0^t s^k \operatorname{Re}(c_1 e^{it \operatorname{Im}(\lambda_1) - \lambda_1 s}) dW(s) \xrightarrow{\text{a.s.}} 0 \quad \text{as } t \rightarrow \infty$$

by the strong law of martingales, see, e.g., Liptser and Shiryaev [8, Chapter 2, §6, Theorem 10]. Obviously, $I_3(t) \xrightarrow{\text{a.s.}} 0$ as $t \rightarrow \infty$, hence we obtain (4.9).

In order to prove (4.10), put

$$V_\ell(t) := \operatorname{Re}(c_\ell U_{\lambda_\ell}^{(\vartheta)} e^{it \operatorname{Im}(\lambda_\ell)}), \quad t \in \mathbb{R}, \quad \ell \in \{1, 2\}.$$

For each $T \in \mathbb{R}_+$, we have

$$\int_0^T e^{-t \operatorname{Re}(\lambda_1 + \lambda_2)} V_1(T-t) V_2(T-t) dt = e^{-T \operatorname{Re}(\lambda_1 + \lambda_2)} \int_0^T e^{t \operatorname{Re}(\lambda_1 + \lambda_2)} V_1(t) V_2(t) dt,$$

hence

$$T^{-\alpha_1 - \alpha_2} e^{-T \operatorname{Re}(\lambda_1 + \lambda_2)} \int_0^T Y_1(t) Y_2(t) dt - \int_0^\infty e^{-t \operatorname{Re}(\lambda_1 + \lambda_2)} V_1(T-t) V_2(T-t) dt$$

$$= J_1(T) + J_2(T) + J_3(T) - J_4(T) - J_5(T)$$

with

$$J_1(T) := T^{-\alpha_1-\alpha_2} e^{-T \operatorname{Re}(\lambda_1+\lambda_2)} \int_0^T [Y_1(t) - t^{\alpha_1} e^{t \operatorname{Re}(\lambda_1)} V_1(t)] [Y_2(t) - t^{\alpha_2} e^{t \operatorname{Re}(\lambda_2)} V_2(t)] dt,$$

$$J_2(T) := T^{-\alpha_1-\alpha_2} e^{-T \operatorname{Re}(\lambda_1+\lambda_2)} \int_0^T [Y_1(t) - t^{\alpha_1} e^{t \operatorname{Re}(\lambda_1)} V_1(t)] t^{\alpha_2} e^{t \operatorname{Re}(\lambda_2)} V_2(t) dt,$$

$$J_3(T) := T^{-\alpha_1-\alpha_2} e^{-T \operatorname{Re}(\lambda_1+\lambda_2)} \int_0^T [Y_2(t) - t^{\alpha_2} e^{t \operatorname{Re}(\lambda_2)} V_2(t)] t^{\alpha_1} e^{t \operatorname{Re}(\lambda_1)} V_1(t) dt,$$

$$J_4(T) := e^{-T \operatorname{Re}(\lambda_1+\lambda_2)} \int_0^T \left(1 - \frac{t^{\alpha_1+\alpha_2}}{T^{\alpha_1+\alpha_2}}\right) e^{t \operatorname{Re}(\lambda_1+\lambda_2)} V_1(t) V_2(t) dt,$$

$$J_5(T) := \int_T^\infty e^{-t \operatorname{Re}(\lambda_1+\lambda_2)} V_1(T-t) V_2(T-t) dt.$$

By (4.9) and L'Hôpital's rule, $J_1(T) \xrightarrow{\text{a.s.}} 0$ as $T \rightarrow \infty$. By the Cauchy-Schwarz inequality, $|J_2(T)| \leq \sqrt{J_6(T) J_7(T)}$, $T \in \mathbb{R}_+$, where, by (4.9) and L'Hôpital's rule,

$$J_6(T) := T^{-2\alpha_1} e^{-2T \operatorname{Re}(\lambda_1)} \int_0^T [Y_1(t) - t^{\alpha_1} e^{t \operatorname{Re}(\lambda_1)} V_1(t)]^2 dt \xrightarrow{\text{a.s.}} 0, \quad \text{as } T \rightarrow \infty,$$

and

$$\begin{aligned} J_7(T) &:= T^{-2\alpha_2} e^{-2T \operatorname{Re}(\lambda_2)} \int_0^T t^{2\alpha_2} e^{2t \operatorname{Re}(\lambda_2)} V_2(t)^2 dt \\ &= \int_0^T \left(1 - \frac{t}{T}\right)^{2\alpha_2} e^{-2t \operatorname{Re}(\lambda_2)} V_2(T-t)^2 dt \leq \frac{1}{2 \operatorname{Re}(\lambda_2)} \sup_{t \in \mathbb{R}} V_2(t)^2 < \infty \quad \text{a.s.}, \end{aligned}$$

since $(V_2(t))_{t \in \mathbb{R}}$ is a continuous and periodic process. Consequently, $J_2(T) \xrightarrow{\text{a.s.}} 0$ as $T \rightarrow \infty$. In a similar way, $J_3(T) \xrightarrow{\text{a.s.}} 0$ as $T \rightarrow \infty$, and

$$\begin{aligned} J_4(T) &\leq \left(\sup_{t \in \mathbb{R}} |V_1(t) V_2(t)| \right) e^{-T \operatorname{Re}(\lambda_1+\lambda_2)} \int_0^T \left(1 - \frac{t^{\alpha_1+\alpha_2}}{T^{\alpha_1+\alpha_2}}\right) e^{t \operatorname{Re}(\lambda_1+\lambda_2)} dt \\ &= \left(\sup_{t \in \mathbb{R}} |V_1(t) V_2(t)| \right) \int_0^T \left[1 - \left(1 - \frac{t}{T}\right)^{\alpha_1+\alpha_2}\right] e^{-t \operatorname{Re}(\lambda_1+\lambda_2)} dt \xrightarrow{\text{a.s.}} 0 \end{aligned}$$

as $T \rightarrow \infty$ by the dominated convergence theorem. Finally,

$$|J_5(T)| \leq \frac{e^{-T \operatorname{Re}(\lambda_1+\lambda_2)}}{\operatorname{Re}(\lambda_1 + \lambda_2)} \sup_{t \in \mathbb{R}} |V_1(t) V_2(t)| \xrightarrow{\text{a.s.}} 0 \quad \text{as } T \rightarrow \infty,$$

hence we obtain (4.10). \square

Proof of Theorem 3.1. The continuous process $(Y^{(\vartheta)}(t))_{t \in \mathbb{R}_+}$ admits the representation (4.1) on $[r, \infty)$. The aim of the following discussion is to show that the function $(y_\vartheta(t))_{t \in \mathbb{R}_+}$ is square integrable. Let $c \in (v_\vartheta^*, 0)$, and apply the representation (4.2). By the definition of

v_ϑ^* , we obtain $P_{\vartheta,\lambda} = 0$ for each $\lambda \in \Lambda_\vartheta$ with $\operatorname{Re}(\lambda) > v_\vartheta^*$, and hence for each $\lambda \in \Lambda_\vartheta$ with $\operatorname{Re}(\lambda) \geq c$. Thus the representation (4.2) gives $y_\vartheta(t) = o(e^{ct})$ as $t \rightarrow \infty$. Since $(y_\vartheta(t))_{t \in \mathbb{R}_+}$ is continuous, the function $(e^{-ct}y_\vartheta(t))_{t \in \mathbb{R}_+}$ is bounded, implying $\int_0^\infty y_\vartheta(t)^2 dt < \infty$. Hence we can apply Lemma 4.2 to obtain

$$J_{\vartheta,T} = \frac{1}{T} \int_0^T Y^{(\vartheta)}(t)^2 dt = \frac{1}{T} \int_0^r Y^{(\vartheta)}(t)^2 dt + \frac{1}{T} \int_r^T Y^{(\vartheta)}(t)^2 dt \xrightarrow{\mathbb{P}} \int_0^\infty y_\vartheta(t)^2 dt = J_\vartheta$$

as $T \rightarrow \infty$, where $J_\vartheta > 0$, since $x_{0,\vartheta} \neq 0$ and $a \neq 0$ implies $y_\vartheta \neq 0$.

In case of $\vartheta = 0$, we have $h_0(\lambda) = \lambda$, $\Lambda_0 = \{0\}$, $m_0(0) = 1$ and $P_{0,0}(t) = A_{0,-1}(0) \int_{[-r,0]} a(du) = a([-r,0])$, $t \in \mathbb{R}$, since $1/h_0(z) = z^{-1}$ yields $A_{0,-1}(0) = 1$. The assumption $a([-r,0]) = 0$ implies $P_{0,0} = 0$, and hence $v_0^* = -\infty$ and $m_0^* = -\infty$. Moreover, the assumption $a([-r,0]) = 0$ yields

$$y_0(t) = \int_{[-r,0]} x_{0,0}(t+u) a(du) = \begin{cases} 0 & \text{if } t \in [r, \infty), \\ a([-t, 0]) & \text{if } t \in [0, r], \end{cases}$$

and we obtain the formula for J_0 .

Further, the process

$$M^{(\vartheta)}(T) := \int_0^T Y^{(\vartheta)}(t) dW(t), \quad T \in \mathbb{R}_+,$$

is a continuous martingale with $M^{(\vartheta)}(0) = 0$ and with quadratic variation

$$\langle M^{(\vartheta)} \rangle(T) = \int_0^T Y^{(\vartheta)}(t)^2 dt,$$

hence Theorem VIII.5.42 of Jacod and Shiryaev [6] yields the statement. \square

Proof of Theorem 3.2. For each $T \in \mathbb{R}_{++}$, we have

$$\Delta_{\vartheta,T} = \frac{1}{T^{m_\vartheta^*+1}} \int_0^T Y^{(\vartheta)}(t) dW(t), \quad J_{\vartheta,T} = \frac{1}{T^{2(m_\vartheta^*+1)}} \int_0^T Y^{(\vartheta)}(t)^2 dt.$$

The continuous process $(Y^{(\vartheta)}(t))_{t \in \mathbb{R}_+}$ admits the representation (4.1) on $[r, \infty)$. We choose $c < 0$ with $c > \sup\{\operatorname{Re}(\lambda) : \lambda \in \Lambda_\vartheta, \operatorname{Re}(\lambda) < 0\}$, and apply the representation (4.2). The assumption $v_\vartheta^* = 0$ yields that $P_{\vartheta,\lambda} = 0$ for each $\lambda \in \Lambda_\vartheta$ with $\operatorname{Re}(\lambda) > 0$, hence we obtain

$$(4.11) \quad y_\vartheta(t) = \sum_{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R})} P_{\vartheta,\lambda}(t) e^{it \operatorname{Im}(\lambda)} + \Psi_{\vartheta,c}(t), \quad t \in \mathbb{R}_+.$$

The leading term of the polynomial $P_{\vartheta,\lambda}$ is $c_{\vartheta,\lambda,m_\vartheta^*} t^{\tilde{m}_\vartheta(\lambda)}$, thus, by the representation (4.1),

$$(4.12) \quad Y^{(\vartheta)}(t) = \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} e^{it \operatorname{Im}(\lambda)} \int_0^t (t-s)^{m_\vartheta^*} e^{-is \operatorname{Im}(\lambda)} dW(s) + \tilde{Y}(t), \quad t \in \mathbb{R}_+,$$

where $(\tilde{Y}(t))_{t \in \mathbb{R}_+}$ is a continuous process satisfying

$$\tilde{Y}(t) := y_\vartheta(t)X_0(0) + \vartheta I_\vartheta(t) + I_\Psi(t) + S(t), \quad t \in [r, \infty),$$

with

$$\begin{aligned} I_\vartheta(t) &:= \int_{[-r, 0]} \int_v^0 y_\vartheta(t+v-s)X_0(s) \, ds \, a(dv), & I_\Psi(t) &:= \int_0^t \Psi_{\vartheta, c}(t-s) \, dW(s), \\ S(t) &:= \sum_{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R})} \sum_{\ell=0}^{\tilde{m}_\vartheta(\lambda) \wedge (m_\vartheta^* - 1)} c_{\vartheta, \lambda, \ell} I_\ell(t), & I_\ell(t) &:= \int_0^t (t-s)^\ell e^{\lambda(t-s)} \, dW(s), \end{aligned}$$

for $t \in [r, \infty)$ and $\ell \in \mathbb{Z}_+$. The aim of the following discussion is to show that $T^{-2(m_\vartheta^*+1)} \int_r^T \tilde{Y}(t)^2 \, dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$. First we show that $T^{-2(m_\vartheta^*+1)} \int_r^T y_\vartheta(t)^2 \, dt \rightarrow 0$ as $T \rightarrow \infty$. For each $\lambda \in \Lambda_\vartheta$ with $\operatorname{Re}(\lambda) = 0$, we have

$$T^{-2(m_\vartheta^*+1)} \int_r^T |P_{\vartheta, \lambda}(t)e^{\lambda t}|^2 \, dt = T^{-2(m_\vartheta^*+1)} \int_r^T |P_{\vartheta, \lambda}(t)|^2 \, dt \rightarrow 0 \quad \text{as } T \rightarrow \infty,$$

since the degree of the complex-valued polynomial $P_{\vartheta, \lambda}$ is $\tilde{m}_\vartheta(\lambda)$, which is at most m_ϑ^* . Moreover,

$$T^{-2(m_\vartheta^*+1)} \int_r^T \Psi_{\vartheta, c}(t)^2 \, dt \leq T^{-2} \int_r^T \Psi_{\vartheta, c}(t)^2 \, dt \leq T^{-2} \int_r^\infty \Psi_{\vartheta, c}(t)^2 \, dt \rightarrow 0$$

as $T \rightarrow \infty$, since $\int_r^\infty |\Psi_{\vartheta, c}(t)|^2 \, dt < \infty$, because the function $(e^{-ct}|\Psi_{\vartheta, c}(t)|)_{t \in \mathbb{R}_+}$ is bounded. Thus, by the representation (4.11), we obtain $T^{-2(m_\vartheta^*+1)} \int_r^T y_\vartheta(t)^2 \, dt \rightarrow 0$ as $T \rightarrow \infty$. Next, by Lemma 4.1,

$$\int_r^T I_\vartheta(t)^2 \, dt \leq r \|a\| \int_{-r}^0 X_0(s)^2 \, ds \int_0^T y_\vartheta(v)^2 \, dv,$$

hence we obtain $T^{-2(m_\vartheta^*+1)} \int_r^T I_\vartheta(t)^2 \, dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$. Further, we have

$$\begin{aligned} (4.13) \quad \mathbb{E} \left(\int_r^T I_\Psi(t)^2 \, dt \right) &= \int_r^T \left(\int_0^t \Psi_{\vartheta, c}(t-s)^2 \, ds \right) \, dt = \int_r^T \left(\int_0^t \Psi_{\vartheta, c}(u)^2 \, du \right) \, dt \\ &= \int_r^T \Psi_{\vartheta, c}(u)^2 \left(\int_u^T \, dt \right) \, du = \int_r^T (T-u) \Psi_{\vartheta, c}(u)^2 \, du \leq T \int_r^T \Psi_{\vartheta, c}(u)^2 \, du, \end{aligned}$$

hence $T^{-2} \int_r^T I_\Psi(t)^2 \, dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$. Finally, for each $\lambda \in \Lambda_\vartheta$ with $\operatorname{Re}(\lambda) = 0$ and each $\ell \in \mathbb{Z}_+$ with $\ell \leq m_\vartheta^* - 1$, we have $T^{-2(m_\vartheta^*+1)} \int_0^T I_\ell(t)^2 \, dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$. Indeed,

$$\begin{aligned} \frac{1}{T^{2(m_\vartheta^*+1)}} \mathbb{E} \left(\int_0^T I_\ell(t)^2 \, dt \right) &= \frac{1}{T^{2(m_\vartheta^*+1)}} \int_0^T \left(\int_0^t (t-s)^{2\ell} \, ds \right) \, dt \\ &= \frac{1}{T^{2(m_\vartheta^*+1)}} \int_0^T \left(\int_0^t u^{2\ell} \, du \right) \, dt = \frac{1}{(2\ell+1)(2\ell+2)T^{2(m_\vartheta^*-\ell)}} \rightarrow 0 \quad \text{as } T \rightarrow \infty. \end{aligned}$$

Hence we conclude $T^{-2(m_\vartheta^*+1)} \int_0^T \tilde{Y}(t)^2 dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$.

Introduce the complex-valued processes $(Z_{\varphi,\ell}(t))_{t \in \mathbb{R}_+}$, $\varphi \in \mathbb{R}$, $\ell \in \mathbb{Z}_+$, by

$$Z_{\varphi,\ell}(t) := \int_0^t (t-s)^\ell e^{-i\varphi s} dW(s).$$

Note that $Z_{0,0} = W$. For each $T \in \mathbb{R}_{++}$, consider the complex-valued processes $(Z_{\varphi,\ell}^T(t))_{s \in [0,1]}$, $\varphi \in \mathbb{R}$, $\ell \in \mathbb{Z}_+$, and the real-valued process $(X^T(s))_{s \in [0,1]}$, defined by

$$Z_{\varphi,\ell}^T(s) := \frac{1}{T^{\ell+\frac{1}{2}}} Z_{\varphi,\ell}(Ts) = \int_0^s (s-u)^\ell e^{-iT\varphi u} dW^T(u),$$

$$X^T(s) := \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} e^{iT s \operatorname{Im}(\lambda)} Z_{\operatorname{Im}(\lambda),m_\vartheta^*}^T(s).$$

Then, for each $T \in \mathbb{R}_{++}$, we have

$$Y^{(\vartheta)}(t) = T^{m_\vartheta^*+\frac{1}{2}} X^T\left(\frac{t}{T}\right) + \tilde{Y}(t), \quad t \in \mathbb{R}_+,$$

and hence,

$$\Delta_{\vartheta,T} = \frac{1}{\sqrt{T}} \int_0^T X^T\left(\frac{t}{T}\right) dW(t) + I_1(T) = \int_0^1 X^T(s) dW^T(s) + I_1(T),$$

$$J_{\vartheta,T} = \frac{1}{T} \int_0^T \left| X^T\left(\frac{t}{T}\right) \right|^2 dt + 2I_2(T) + I_3(T) = \int_0^1 |X^T(s)|^2 ds + 2I_2(T) + I_3(T),$$

with

$$I_1(T) := \frac{1}{T^{m_\vartheta^*+1}} \int_0^T \tilde{Y}(t) dW(t), \quad I_2(T) := \frac{1}{T^{m_\vartheta^*+\frac{3}{2}}} \int_0^T \operatorname{Re}\left(X^T\left(\frac{t}{T}\right) \overline{\tilde{Y}(t)}\right) dt,$$

$$I_3(T) := \frac{1}{T^{2(m_\vartheta^*+1)}} \int_0^T |\tilde{Y}(t)|^2 dt.$$

Introducing the process

$$Y^T(t) := \int_0^t X^T(s) dW^T(s), \quad t \in \mathbb{R}_+, \quad T \in \mathbb{R}_{++},$$

we have

$$\int_0^t X^T(s)^2 ds = [Y^T, Y^T](t), \quad t \in \mathbb{R}_+, \quad T \in \mathbb{R}_{++},$$

where $([U, V](t))_{t \in \mathbb{R}_+}$ denotes the quadratic covariation process of the processes $(U(t))_{t \in \mathbb{R}_+}$ and $(V(t))_{t \in \mathbb{R}_+}$. Moreover,

$$\begin{aligned} Y^T(t) &= \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} \int_0^t e^{iT s \operatorname{Im}(\lambda)} Z_{\operatorname{Im}(\lambda),m_\vartheta^*}^T(s) dW^T(s) \\ &= \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (i\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} \int_0^t Z_{\operatorname{Im}(\lambda),m_\vartheta^*}^T(s) d\overline{Z_{\operatorname{Im}(\lambda),0}^T(s)}. \end{aligned}$$

By the functional central limit theorem,

$$(Z_{\text{Im}(\lambda),0}^T : \lambda \in \Lambda_\vartheta \cap (\text{i}\mathbb{R})) \xrightarrow{\mathcal{D}} (\mathcal{Z}_{\text{Im}(\lambda),0} : \lambda \in \Lambda_\vartheta \cap (\text{i}\mathbb{R})), \quad \text{as } T \rightarrow \infty,$$

since

$$Z_{\varphi,0}^T(s) = \int_0^s \cos(T\varphi u) dW(u) - \text{i} \int_0^s \sin(T\varphi u) dW(u), \quad Z_{-\varphi,0}^T(s) = \overline{Z_{\varphi,0}^T(s)},$$

for all $s \in [0, 1]$, $\varphi \in \mathbb{R}$ and $T \in \mathbb{R}_{++}$, and

$$\begin{aligned} \int_0^s \cos(T\varphi_1 u) \cos(T\varphi_2 u) du &= \frac{1}{2} \int_0^s [\cos(T(\varphi_1 - \varphi_2)u) + \cos(T(\varphi_1 + \varphi_2)u)] du \\ &= \begin{cases} \frac{s}{2} + \frac{\sin(2T\varphi_1 s)}{4T\varphi_1} \rightarrow \frac{s}{2}, & \text{if } \varphi_1 = \varphi_2, \\ \frac{\sin(T(\varphi_1 - \varphi_2)s)}{2T(\varphi_1 - \varphi_2)} + \frac{\sin(T(\varphi_1 + \varphi_2)s)}{2T(\varphi_1 + \varphi_2)} \rightarrow 0, & \text{if } \varphi_1 \neq \varphi_2, \end{cases} \\ \int_0^s \sin(T\varphi_1 u) \sin(T\varphi_2 u) du &= \frac{1}{2} \int_0^s [\cos(T(\varphi_1 - \varphi_2)u) - \cos(T(\varphi_1 + \varphi_2)u)] du \\ &= \begin{cases} \frac{s}{2} - \frac{\sin(2T\varphi_1 s)}{4T\varphi_1} \rightarrow \frac{s}{2}, & \text{if } \varphi_1 = \varphi_2, \\ \frac{\sin(T(\varphi_1 - \varphi_2)s)}{2T(\varphi_1 - \varphi_2)} - \frac{\sin(T(\varphi_1 + \varphi_2)s)}{2T(\varphi_1 + \varphi_2)} \rightarrow 0, & \text{if } \varphi_1 \neq \varphi_2, \end{cases} \\ \int_0^s \sin(T\varphi_1 u) \cos(T\varphi_2 u) du &= \frac{1}{2} \int_0^s [\sin(T(\varphi_1 + \varphi_2)u) + \sin(T(\varphi_1 - \varphi_2)u)] du \\ &= \begin{cases} \frac{1 - \cos(2T\varphi_1 s)}{4T\varphi_1} \rightarrow 0, & \text{if } \varphi_1 = \varphi_2, \\ \frac{1 - \cos(T(\varphi_1 + \varphi_2)s)}{2T(\varphi_1 + \varphi_2)} + \frac{1 - \cos(T(\varphi_1 - \varphi_2)s)}{2T(\varphi_1 - \varphi_2)} \rightarrow 0, & \text{if } \varphi_1 \neq \varphi_2, \end{cases} \end{aligned}$$

as $T \rightarrow \infty$ for all $s \in [0, 1]$ and $\varphi_1, \varphi_2 \in \mathbb{R}_+$. Consequently,

$$(Z_{\text{Im}(\lambda),0}^T, Z_{\text{Im}(\lambda),m_\vartheta}^T : \lambda \in \Lambda_\vartheta \cap (\text{i}\mathbb{R})) \xrightarrow{\mathcal{D}} (\mathcal{Z}_{\text{Im}(\lambda),0}, \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta} : \lambda \in \Lambda_\vartheta \cap (\text{i}\mathbb{R})), \quad \text{as } T \rightarrow \infty,$$

and hence, by Itô's formula and the continuous mapping theorem,

$$Y^T \xrightarrow{\mathcal{D}} \mathcal{Y} \quad \text{as } T \rightarrow \infty$$

with

$$\mathcal{Y}(t) = \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (\text{i}\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} c_{\vartheta,\lambda,m_\vartheta^*} \int_0^t \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s) d\overline{\mathcal{Z}_{\text{Im}(\lambda),0}(s)}.$$

Further, by Corollary 4.12 in Gushchin and K  chler [3],

$$(Y^T(1), [Y^T, Y^T](1)) \xrightarrow{\mathcal{D}} (\mathcal{Y}(1), [\mathcal{Y}, \mathcal{Y}](1)) \quad \text{as } T \rightarrow \infty.$$

We also have

$$\begin{aligned}
\mathcal{Y}(t) &= \mathbb{1}_{\{0 \in \Lambda_\vartheta, \tilde{m}_\vartheta(0) = m_\vartheta^*\}} c_{\vartheta,0,m_\vartheta^*} \int_0^t \mathcal{Z}_{0,m_\vartheta^*}(s) d\mathcal{W}(s) \\
&\quad + \frac{1}{\sqrt{2}} \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (\mathbb{i}\mathbb{R}_{++}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} \left[\int_0^t \left(c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s) + \overline{c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s)} \right) d\mathcal{W}_{\text{Im}(\lambda),\text{Re}}(s) \right. \\
&\quad \left. + \int_0^t \left(-c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s) + \overline{c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s)} \right) d\mathcal{W}_{\text{Im}(\lambda),\text{Im}}(s) \right] \\
&= \mathbb{1}_{\{0 \in \Lambda_\vartheta, \tilde{m}_\vartheta(0) = m_\vartheta^*\}} c_{\vartheta,0,m_\vartheta^*} \int_0^t \mathcal{Z}_{0,m_\vartheta^*}(s) d\mathcal{W}(s) \\
&\quad + \sqrt{2} \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (\mathbb{i}\mathbb{R}_{++}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} \left[\int_0^t \text{Re}(c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s)) d\mathcal{W}_{\text{Im}(\lambda),\text{Re}}(s) \right. \\
&\quad \left. - \int_0^t \text{Im}(c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s)) d\mathcal{W}_{\text{Im}(\lambda),\text{Im}}(s) \right],
\end{aligned}$$

hence

$$\begin{aligned}
[\mathcal{Y}, \mathcal{Y}](1) &= \mathbb{1}_{\{0 \in \Lambda_\vartheta, \tilde{m}_\vartheta(0) = m_\vartheta^*\}} c_{\vartheta,0,m_\vartheta^*}^2 \int_0^1 \mathcal{Z}_{0,m_\vartheta^*}(s)^2 ds \\
&\quad + 2 \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (\mathbb{i}\mathbb{R}_{++}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} \left[\int_0^1 \text{Re}(c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s))^2 ds + \int_0^1 \text{Im}(c_{\vartheta,\lambda,m_\vartheta^*} \mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s))^2 ds \right] \\
&= \sum_{\substack{\lambda \in \Lambda_\vartheta \cap (\mathbb{i}\mathbb{R}) \\ \tilde{m}_\vartheta(\lambda) = m_\vartheta^*}} |c_{\vartheta,\lambda,m_\vartheta^*}|^2 \int_0^1 |\mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s)|^2 ds.
\end{aligned}$$

Recall that $I_3(T) \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$, which also implies $I_1(T) \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$. Further,

$$|I_2(T)| \leq \sqrt{\frac{1}{T^{2\ell+3}} \int_0^T \left| X^T\left(\frac{t}{T}\right) \right|^2 dt \int_0^T |\tilde{Y}(t)|^2 dt} = \sqrt{\int_0^1 |X^T(s)|^2 ds \frac{1}{T^2} \int_0^T |\tilde{Y}(t)|^2 dt} \xrightarrow{\mathbb{P}} 0$$

as $T \rightarrow \infty$, hence we obtain

$$(\Delta_{\vartheta,T}, J_{\vartheta,T}) \xrightarrow{\mathcal{D}} (\Delta_\vartheta, J_\vartheta) \quad \text{as } T \rightarrow \infty.$$

Moreover, we have $J_\vartheta > 0$ almost surely. Indeed, $J_\vartheta = 0$ would imply $\int_0^1 |\mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s)|^2 ds = 0$ for all $\lambda \in \{\lambda \in \Lambda_\vartheta \cap (\mathbb{i}\mathbb{R}) : \tilde{m}_\vartheta(\lambda) = m_\vartheta^*\}$, which, in turn, would imply $\mathcal{Z}_{\text{Im}(\lambda),m_\vartheta^*}(s) = 0$ for

all $s \in [0, 1]$. But this is in a contradiction with the fact that $\mathcal{Z}_{\text{Im}(\lambda), m_\vartheta^*}$ is a non-degenerate Gaussian process. \square

Proof of Theorem 3.3. We have

$$J_{\vartheta, T} = T^{-2m_\vartheta^*} e^{-2v_\vartheta^* T} \int_0^T Y^{(\vartheta)}(t)^2 dt, \quad T \in \mathbb{R}_+.$$

The process $(Y^{(\vartheta)}(t))_{t \in [r, \infty)}$ admits the representation (4.1). We choose $c < v_\vartheta^*$ with $c > \sup\{\text{Re}(\lambda) : \lambda \in \Lambda_\vartheta, \text{Re}(\lambda) < v_\vartheta^*\}$, and apply the representation (4.2). By the definition of v_ϑ^* , we obtain $P_{\vartheta, \lambda} = 0$ for each $\lambda \in \Lambda_\vartheta$ with $\text{Re}(\lambda) > v_\vartheta^*$, hence we obtain

$$(4.14) \quad y_\vartheta(t) = \sum_{\lambda \in \Lambda_\vartheta \cap (v_\vartheta^* + i\mathbb{R})} \sum_{\ell=0}^{\tilde{m}_\vartheta(v_\vartheta^*)} c_{\vartheta, \lambda, \ell} t^\ell e^{\lambda t} + \Psi_{\vartheta, c}(t), \quad t \in \mathbb{R}_+.$$

For each $\lambda \in \Lambda_\vartheta$ and $\ell \in \{0, \dots, \tilde{m}_\vartheta(\lambda)\}$, we have

$$\begin{aligned} c_{\vartheta, \lambda, \ell} t^\ell e^{\lambda t} + c_{\vartheta, \bar{\lambda}, \ell} t^\ell e^{\bar{\lambda} t} &= t^\ell (c_{\vartheta, \lambda, \ell} e^{\lambda t} + \overline{c_{\vartheta, \lambda, \ell} e^{\lambda t}}) \\ &= 2t^\ell \text{Re}(c_{\vartheta, \lambda, \ell} e^{\lambda t}) = t^\ell [\text{Re}(c_{\vartheta, \lambda, \ell} e^{\lambda t}) + \text{Re}(c_{\vartheta, \bar{\lambda}, \ell} e^{\bar{\lambda} t})], \end{aligned}$$

hence (4.14) can also be written in the form

$$y_\vartheta(t) = \sum_{\lambda \in \Lambda_\vartheta \cap (v_\vartheta^* + i\mathbb{R})} \sum_{\ell=0}^{\tilde{m}_\vartheta(\lambda)} t^\ell \text{Re}(c_{\vartheta, \lambda, \ell} e^{\lambda t}) + \Psi_{\vartheta, c}(t), \quad t \in \mathbb{R}_+.$$

Consequently, by the representation (4.1), we have

$$(4.15) \quad Y^{(\vartheta)}(t) = Y(t) + \tilde{Y}(t), \quad t \in \mathbb{R}_+,$$

with

$$Y(t) := \sum_{\lambda \in \Lambda_\vartheta \cap (v_\vartheta^* + i\mathbb{R})} \sum_{\ell=0}^{\tilde{m}_\vartheta(\lambda)} Y_{\vartheta, \lambda, \ell}(t), \quad t \in \mathbb{R}_+,$$

where the continuous processes $(Y_{\vartheta, \lambda, \ell}(t))_{t \in \mathbb{R}_+}$, $\lambda \in \Lambda$, $\ell \in \{0, \dots, \tilde{m}_\vartheta(v_\vartheta^*)\}$, and $(\tilde{Y}(t))_{t \in \mathbb{R}_+}$ admit representation (4.5) on $[r, \infty)$ with $y(t) = t^\ell \text{Re}(c_{\vartheta, \lambda, \ell} e^{\lambda t})$, $\lambda \in \Lambda$, $\ell \in \{0, \dots, \tilde{m}_\vartheta(v_\vartheta^*)\}$, and with $y(t) = \Psi_{\vartheta, c}(t)$, $t \in \mathbb{R}_+$, respectively. The aim of the following discussion is to show that $T^{-2m_\vartheta^*} e^{-2v_\vartheta^* T} \int_0^T \tilde{Y}(t)^2 dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$. We have

$$\tilde{Y}(t) := \Psi_{\vartheta, c}(t) X_0(0) + \vartheta I_\vartheta(t) + I_\Psi(t), \quad t \in [r, \infty),$$

with

$$I_\vartheta(t) := \int_{[-r, 0]} \int_v^0 \Psi_{\vartheta, c}(t + v - s) X_0(s) ds a(dv), \quad I_\Psi(t) := \int_0^t \Psi_{\vartheta, c}(t - s) dW(s).$$

The function $(e^{-ct}\Psi_{\vartheta,c}(t))_{t \in \mathbb{R}_+}$ is bounded, hence $|\Psi_{\vartheta,c}(t)| \leq Ce^{ct}$ for all $t \in \mathbb{R}_+$ with $C := \sup_{t \in \mathbb{R}_+} e^{-ct}|\Psi_{\vartheta,c}(t)| < \infty$, hence

$$T^{-2m_{\vartheta}^*}e^{-2v_{\vartheta}^*T} \int_r^T \Psi_{\vartheta,c}(t)^2 dt \leq e^{-2v_{\vartheta}^*T} \int_0^T C^2 e^{2ct} dt = \frac{C^2}{2c} e^{-2(v_{\vartheta}^*-c)T} \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Next, by Lemma 4.1,

$$\int_r^T I_{\vartheta}(t)^2 dt \leq r \|a\| \int_{-r}^0 X_0(s)^2 ds \int_0^T \psi_{\vartheta}(v)^2 dv,$$

hence we obtain $T^{-2m_{\vartheta}^*}e^{-2v_{\vartheta}^*T} \int_0^T I_{\vartheta}(t)^2 dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$. Finally, by (4.13), $T^{-2m_{\vartheta}^*}e^{-2v_{\vartheta}^*T} \int_0^T I_{\Psi}(t)^2 dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$, and we conclude $T^{-2m_{\vartheta}^*}e^{-2v_{\vartheta}^*T} \int_0^T \tilde{Y}(t)^2 dt \xrightarrow{\mathbb{P}} 0$ as $T \rightarrow \infty$.

Applying Lemma 4.3, we obtain

$$\begin{aligned} & T^{-\ell_1-\ell_2}e^{-2v_{\vartheta}^*T} \int_0^T Y_{\vartheta,\lambda_1,\ell_1}(t)Y_{\vartheta,\lambda_2,\ell_2}(t) dt \\ & - \int_0^{\infty} e^{-2v_{\vartheta}^*t} \operatorname{Re}(c_{\vartheta,\lambda_1,\ell_1}U_{\lambda_1}^{(\vartheta)}e^{i(T-t)\operatorname{Im}(\lambda_1)}) \operatorname{Re}(c_{\vartheta,\lambda_2,\ell_2}U_{\lambda_2}^{(\vartheta)}e^{i(T-t)\operatorname{Im}(\lambda_2)}) dt \xrightarrow{\text{a.s.}} 0 \end{aligned}$$

as $T \rightarrow \infty$ for each $\lambda_1, \lambda_2 \in \Lambda$ with $\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2) = v_{\vartheta}^*$ and $\ell_1 \in \{0, \dots, \tilde{m}_{\vartheta}(\lambda_1)\}$, $\ell_2 \in \{0, \dots, \tilde{m}_{\vartheta}(\lambda_2)\}$. Consequently,

$$T^{-2m_{\vartheta}^*}e^{-2v_{\vartheta}^*T} \int_0^T Y(t)^2 dt - \int_0^{\infty} e^{-2v_{\vartheta}^*t} \left(\sum_{\lambda \in \Lambda_{\vartheta} \cap (v_{\vartheta}^* + i\mathbb{R})} \operatorname{Re}(c_{\vartheta,\lambda,m_{\vartheta}^*}U_{\lambda}^{(\vartheta)}e^{i(T-t)\operatorname{Im}(\lambda)}) \right)^2 dt \xrightarrow{\text{a.s.}} 0$$

as $T \rightarrow \infty$, hence we obtain $J_{\vartheta,T} - J_{\vartheta}(T) \xrightarrow{\text{a.s.}} 0$ as $T \rightarrow \infty$. Since $H_{\vartheta} \neq \emptyset$ and the numbers in H_{ϑ} have a common divisor D_{ϑ} , the process $(J_{\vartheta}(t))_{t \in \mathbb{R}_+}$ is periodic with period $\frac{2\pi}{D_{\vartheta}}$, and, by Theorem VIII.5.42 of Jacod and Shiryaev [6], we conclude

$$(\Delta_{\vartheta,kD+d}, J_{\vartheta,kD+d}) \xrightarrow{\mathcal{D}} (\Delta_{\vartheta}(d), J_{\vartheta}(d)) \quad \text{as } k \rightarrow \infty$$

for all $d \in [0, \frac{2\pi}{D_{\vartheta}})$. Moreover, we have $J_{\vartheta}(d) > 0$ almost surely for all $d \in [0, \frac{2\pi}{D_{\vartheta}})$. Indeed, if $J_{\vartheta}(d) = 0$ almost surely for all $d \in [0, \frac{2\pi}{D_{\vartheta}})$, then

$$\operatorname{Re} \left(\sum_{\substack{\lambda \in \Lambda_{\vartheta} \cap (v_{\vartheta}^* + i\mathbb{R}) \\ \tilde{m}_{\vartheta}(\lambda) = m_{\vartheta}^*}} c_{\vartheta,\lambda,m_{\vartheta}^*} U_{\lambda}^{(\vartheta)} e^{i(d-t)\operatorname{Im}(\lambda)} \right) = 0$$

for all $d \in [0, \frac{2\pi}{D_{\vartheta}})$ and $t \in \mathbb{R}_{++}$. But this is in a contradiction with the fact that the left-hand side is a Gaussian random variable with variance

$$\int_0^{\infty} \operatorname{Re} \left(\sum_{\lambda \in \Lambda_{\vartheta} \cap (v_{\vartheta}^* + i\mathbb{R})} c_{\vartheta,\lambda,m_{\vartheta}^*} e^{i(d-t)\operatorname{Im}(\lambda)} e^{-\lambda s} \right)^2 ds \neq 0.$$

Consequently we obtain that the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ is PLAMN at ϑ .

If $H_\vartheta = \emptyset$, then $\Lambda_\vartheta \cap (v_\vartheta^* + i\mathbb{R}) = \{v_\vartheta^*\}$ and $\tilde{m}_\vartheta(v_\vartheta^*) = m_\vartheta^*$, thus

$$T^{-2m_\vartheta^*} e^{-2v_\vartheta^* T} \int_0^T Y(t)^2 dt \xrightarrow{\text{a.s.}} \int_0^\infty e^{-2v_\vartheta^* t} (c_{\vartheta, v_\vartheta^*, m_\vartheta^*} U_\lambda^{(\vartheta)})^2 dt = J_\vartheta, \quad \text{as } T \rightarrow \infty.$$

By Theorem VIII.5.42 of Jacod and Shiryaev [6], we conclude

$$(\Delta_{\vartheta, T}, J_{\vartheta, T}) \xrightarrow{\mathcal{D}} (\Delta_\vartheta, J_\vartheta) \quad \text{as } T \rightarrow \infty,$$

and we obtain that the family $(\mathcal{E}_T)_{T \in \mathbb{R}_{++}}$ is LAMN at ϑ . □

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